Rutgers University: Algebra Written Qualifying Exam

January 2019: Problem 5 Solution

Exercise. Let R be an associative ring with identity. Assume that R has no proper one-sided ideals. Prove that R is a skew-field.

Solution.

A **skew field** is a division ring; i.e. a ring with multiplicative inverses.

Let $I = \{ra : r\}$ for some a.

Then, since I cannot be a proper ideal, $I = R$.

$\implies \exists r \in R$ s.t. $ra = 1$

$\implies r(ar) = (ra)r = 1r$

$\implies r - r(ar) = 0$

$\implies r(1 - ar) = 0$

Claim: R has no zero divisors. (if $xy = 0$ then $x = 0$ or $y = 0$.)

Let $x \in R$, $x \neq 0$

Then Rx is a one-sided ideal, so $Rx = R$.

$\implies \exists s \in R$ s.t. $sx = 1$.

Now suppose $xy = 0$.

$s(xy) = s0 = 0$

$(sx)y = 1y = y$

and $(sx)y = s(xy)$

$\implies y = 0$

Thus, R has no zero divisors.

$\implies r = 0$ or $1 - ar = 0$

But $r \neq 0$ since $ra = 1$

$\implies 1 - ar = 0$

$\implies ar = 1$

Thus, $\forall a \in R$, $\exists r \in R$ s.t. $ar = ra = 1$.

So, R is a skew field.